Further mathematics
Higher level
Paper 1

Wednesday 10 May 2017 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 10]

The mean weight of a certain breed of bird is claimed to be 5.5 kg. In order to test this claim, a random sample of 10 birds of the breed was obtained and weighed, with the following results in kg.

$$5.41 \quad 5.22 \quad 5.54 \quad 5.58 \quad 5.20 \quad 5.57 \quad 5.23 \quad 5.32 \quad 5.46 \quad 5.37$$

You may assume that the weights of this breed of bird are normally distributed.

(a) State suitable hypotheses for testing the above claim using a two-tailed test. [1]

(b) Calculate unbiased estimates of the mean and the variance of the weights of this breed of bird. [4]

(c) (i) Determine the p-value of the above data.

(ii) State whether or not the claim is supported by the data, using a significance level of 5%. [5]

2. [Maximum mark: 7]

(a) Consider the linear congruence $ax \equiv b(\text{mod } p)$ where $a, b \in \mathbb{Z}^+, p$ is a prime and $\gcd(a, p) = 1$. Using Fermat's little theorem, show that $x \equiv a^{p-2}b(\text{mod } p)$. [3]

(b) **Hence** find the smallest value of x greater than 100 satisfying the linear congruence $3x \equiv 13(\text{mod } 19)$. [4]

3. [Maximum mark: 10]

Consider the system of equations

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 1 \\ 5 & 1 & 8 & 0 \\ 3 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ \lambda \\ \mu \end{bmatrix}.$$

(a) Determine the value of λ and the value of μ for which the equations are consistent. [5]

(b) For these values of λ and μ, solve the equations. [3]

(c) State the rank of the matrix of coefficients, justifying your answer. [2]

4. [Maximum mark: 11]

The weights of male students in a college are modelled by a normal distribution with mean $80\,\text{kg}$ and standard deviation $7\,\text{kg}$.
The weights of female students in the college are modelled by a normal distribution with mean $54\,\text{kg}$ and standard deviation $5\,\text{kg}$.

(a) Find the probability that the weight of a randomly chosen male student is more than twice the weight of a randomly chosen female student. [6]

The college has a lift installed with a recommended maximum load of $550\,\text{kg}$. One morning, the lift contains 3 male students and 6 female students. You may assume that the 9 students are randomly chosen.

(b) Determine the probability that their combined weight exceeds the recommended maximum. [5]

5. [Maximum mark: 6]

(a) Given that the series $\sum_{n=1}^{\infty} u_n$ is convergent, where $u_n > 0$, show that the series $\sum_{n=1}^{\infty} u_n^2$ is also convergent. [4]

(b) (i) State the converse proposition.

(ii) By giving a suitable example, show that it is false. [2]

6. [Maximum mark: 8]

The permutation P is given by

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}.$$

(a) Determine the order of P, justifying your answer. [2]

(b) Find P^2. [2]

(c) The permutation group G is generated by P. Determine the element of G that is of order 2, giving your answer in cycle notation. [4]

7. [Maximum mark: 13]

The function f is defined by

$$f(x) = \frac{e^x + e^{-x} + 2\cos x}{4}, \; x \in \mathbb{R}.$$

(a) (i) Show that $f^{(4)}(x) = f(x)$;

(ii) By considering derivatives of f, determine the first three non-zero terms of the Maclaurin series for $f(x)$. [8]

(b) The random variable X has a Poisson distribution with mean μ.

(i) Write down a series in terms of μ for the probability $p = P[X \equiv 0 (\text{mod } 4)]$.

(ii) Show that $p = e^{-\mu} f(\mu)$.

(iii) Determine the numerical value of p when $\mu = 3$. [5]

8. [Maximum mark: 12]

The normal at the point $\mathrm{T}(at^2, 2at)$, $t \neq 0$, on the parabola $y^2 = 4ax$ meets the parabola again at the point $\mathrm{S}(as^2, 2as)$.

(a) Show that $t^2 + st + 2 = 0$. [7]

(b) Given that $\mathrm{S\hat{O}T}$ is a right-angle, where O is the origin, determine the possible values of t. [5]

9. [Maximum mark: 13]

(a) Using l'Hôpital's rule, show that
$\lim\limits_{x \to \infty} x^n \mathrm{e}^{-x} = 0$ where $n \in \mathbb{N}$. [4]

(b) Let
$I_n = \int_1^{\infty} x^n \mathrm{e}^{-x} \mathrm{d}x$ where $n \in \mathbb{N}$.

(i) Show that, for $n \in \mathbb{Z}^+$,
$$I_n = \alpha \mathrm{e}^{-1} + \beta n I_{n-1}$$
where α, β are constants to be determined.

(ii) Determine the value of I_3, giving your answer as a multiple of e^{-1}. [9]

10. [Maximum mark: 9]

Let G denote the set of 2×2 matrices whose elements belong to $\mathbb{R}$ and whose determinant is equal to 1. Let $*$ denote matrix multiplication which may be assumed to be associative.

(a) Show that $\{G, *\}$ is a group. [5]

Let H denote the set of 2×2 matrices whose elements belong to $\mathbb{Z}$ and whose determinant is equal to 1.

(b) Determine whether or not $\{H, *\}$ is a subgroup of $\{G, *\}$. [4]

11. [Maximum mark: 12]

The simple connected planar graph J has the following adjacency table.

	A	B	C	D	E	F	G	H
A	–	0	1	1	1	0	0	0
B	0	–	1	1	1	1	1	1
C	1	1	–	1	1	0	1	1
D	1	1	1	–	1	0	0	0
E	1	1	1	1	–	1	1	0
F	0	1	0	0	1	–	1	0
G	0	1	1	0	1	1	–	1
H	0	1	1	0	0	0	1	–

(a) Without attempting to draw J,

(i) verify that J satisfies the handshaking lemma;

(ii) determine the number of faces in J. [4]

(b) The vertices D and G are joined by a single edge to form the graph K. Show that K is not planar. [3]

(c) (i) Explain why a graph containing a cycle of length three cannot be bipartite.

(ii) Hence by finding a cycle of length three, show that the complement of K is not bipartite. [5]

12. [Maximum mark: 12]

The vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}$ of an acute angled triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ with respect to an origin O.

(a) The mid-point of $[\mathrm{BC}]$ is denoted by D. The point E lies on $[\mathrm{AD}]$ such that $\mathrm{AE} = 2\mathrm{DE}$.

(i) Show that the position vector of E is

$$\frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}).$$

(ii) Explain briefly why this result shows that the three medians of a triangle are concurrent. [5]

(b) The perpendiculars from B to $[\mathrm{AC}]$ and C to $[\mathrm{AB}]$ meet at the point F.

(i) Show that the position vector $\boldsymbol{f}$ of F satisfies the equations

$$(\boldsymbol{b}-\boldsymbol{f})\cdot(\boldsymbol{c}-\boldsymbol{a}) = 0$$
$$(\boldsymbol{c}-\boldsymbol{f})\cdot(\boldsymbol{a}-\boldsymbol{b}) = 0.$$

(ii) Show, by expanding these equations, that

$$(\boldsymbol{a}-\boldsymbol{f})\cdot(\boldsymbol{c}-\boldsymbol{b}) = 0.$$

(iii) Explain briefly why this result shows that the three altitudes of a triangle are concurrent. [7]

13. [Maximum mark: 7]

A random sample $X_1, X_2, \ldots, X_n$ is taken from the normal distribution $\mathrm{N}(\mu, \sigma^2)$, where the value of μ is unknown but the value of σ^2 is known. The mean of the sample is denoted by $\bar{X}$.

(a) (i) State the distribution of $\dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$.

(ii) Hence show that, with probability 0.95,

$$\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}.$$ [5]

(b) A mathematics teacher, wishing to apply the above result, generates some artificial data, assumes a value for the variance and calculates the following 95% confidence interval for μ,

$$[3.12, 3.25].$$

The teacher asks Alun to interpret this result. Alun makes the following statement.
"The value of μ lies in the interval $[3.12, 3.25]$ with probability 0.95."

(i) Explain briefly why this is an incorrect statement.

(ii) Give a correct interpretation. [2]

14. [Maximum mark: 9]

(a) By writing $10 = 11 - 1$, use the binomial theorem to show that

$$10^n \equiv (-1)^n (\mathrm{mod}\ 11) \text{ for } n \in \mathbb{N}.$$ [3]

A number is called palindromic if it reads the same backwards as forwards, for example 524425.

(b) (i) Show that all palindromic numbers to base 10 with an even number of digits are divisible by 11.

(ii) By finding a suitable example, show that this is not necessarily true for palindromic numbers to base 10 with an odd number of digits. [6]

15. [Maximum mark: 11]

(a) The non-zero vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form an orthogonal set of vectors in $\mathbb{R}^3$.

(i) By considering $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 = \boldsymbol{0}$, show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.

(ii) Explain briefly why $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis for vectors in $\mathbb{R}^3$. [6]

(b) (i) Show that the vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix};\begin{bmatrix}-1\\1\\1\end{bmatrix};\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

form an orthogonal basis.

(ii) Express the vector

$$\begin{bmatrix}2\\8\\0\end{bmatrix}$$

as a linear combination of these vectors. [5]